

Periodic Benjamin-Ono equation with discrete Laplacian and 2D-Toda Hierarchy

Jun'ichi Shiraishi

*Graduate School of Mathematical Sciences, The University of Tokyo,
3-8-1 Komaba Meguro-ku Tokyo 153-8914, Japan
E-mail: shiraish@ms.u-tokyo.ac.jp*

Yohei Tutiya

*Kanagawa Institute of Technology,
1030 Shimo-Ogino Atsugi-city Kanagawa 243-0292, Japan
E-mail: tutiya@gen.kanagawa-it.ac.jp*

We study the relation between the periodic Benjamin-Ono equation with discrete Laplacian and the two dimensional Toda hierarchy. We introduce the tau-functions $\tau_{\pm}(z)$ for the periodic Benjamin-Ono equation, construct two families of integrals of motion $\{M_1, M_2, \dots\}$, $\{\overline{M}_1, \overline{M}_2, \dots\}$, and calculate some examples of the bilinear equations using the Hamiltonian structure. We confirmed that some of the low lying bilinear equations agree with the ones obtained from a certain reduction of the 2D Toda hierarchy.

Keywords: Benjamin-Ono equation, 2D Toda hierarchy

1. Introduction

1.1. Periodic Benjamin-Ono equation with discrete Laplacian

Let γ be a complex parameter satisfying $\text{Im}(\gamma) \geq 0$. Let x and t be real independent variables, and $\eta(x, t)$ be an analytic function satisfying the periodicity condition $\eta(x+1, t) = \eta(x, t)$. In Refs. 1 and 2, we considered the integro-differential equation

$$\frac{\partial}{\partial t} \eta(x, t) = \eta(x, t) \cdot \frac{i}{2} \int_{-1/2}^{1/2} (\Delta_{\gamma} \cot(\pi(y-x))) \eta(y, t) dy, \quad (1)$$

where the discrete Laplacian Δ_{γ} is defined by $(\Delta_{\gamma} f)(x) = f(x-\gamma) - 2f(x) + f(x+\gamma)$, and the integral \int means the Cauchy principal value. This can be

regarded as a periodic version of the Benjamin-Ono^{3,4} equation associated with the discrete Laplacian Δ_γ .

For the sake of simplicity, set $z = e^{2\pi ix}$ and $q = e^{2\pi i\gamma}$. By abuse of notation, we use the notation $\eta(z)$ to indicate the dependence on z . From the spatial periodicity, we have the Fourier series expansion $\eta(z) = \sum_{n \in \mathbf{Z}} \eta_n z^{-n}$. Set $\eta_+(z) = \sum_{n > 0} \eta_{-n} z^n$ and $\eta_-(z) = \sum_{n < 0} \eta_n z^{-n}$. Note that we have $\eta(z) = \eta_+(z) + \eta_0 + \eta_-(z)$. Then the equation (1) can be expressed as

$$\begin{aligned} \partial_t \eta(z) &= \eta(z) \sum_{l \neq 0} \operatorname{sgn}(l) (1 - q^{|l|}) \eta_{-l} z^l \\ &= \eta(z) (\eta_+(z) - \eta_+(zq) - \eta_-(z) + \eta_-(z/q)). \end{aligned} \quad (2)$$

1.2. Poisson algebra and Toda field equation

We show that one can introduce another time \bar{t} and obtain the 2D Toda field equation⁵ by using the Poisson Heisenberg algebra for our periodic Benjamin-Ono equation with discrete Laplacian.^{1,2} As for the Hamiltonian structure for the usual Benjamin-Ono equation, see Refs. 6 and 7. See Ref. 8 also.

Our Poisson algebra is generated by α_n ($n \in \mathbf{Z}_{\neq 0}$) with the Poisson brackets

$$\{\alpha_n, \alpha_m\} = \operatorname{sgn}(n) (1 - q^{|n|}) \delta_{n+m,0}, \quad (3)$$

where $\operatorname{sgn}(n) = |n|/n$ for $n \neq 0$ and $\operatorname{sgn}(0) = 0$.

Definition 1.1. Set

$$\tau_+(z) = \exp \left(- \sum_{n>0} \frac{\alpha_{-n}}{1-q^n} z^n \right), \quad \tau_-(z) = \exp \left(- \sum_{n>0} \frac{\alpha_n}{1-q^n} z^{-n} \right). \quad (4)$$

We call $\tau_\pm(z)$ the tau-functions.

Express the dependent variable $\eta(z)$ in terms of the tau-functions and a constant ε as

$$\eta(z) = \sum_{n \in \mathbf{Z}} \eta_n z^{-n} = \varepsilon \exp \left(\sum_{n \neq 0} \alpha_n z^{-n} \right) = \varepsilon \frac{\tau_-(z/q) \tau_+(zq)}{\tau_-(z) \tau_+(z)}. \quad (5)$$

We need to introduce another dependent variables $\xi(z)$ by

$$\begin{aligned}\xi(z) &= \sum_{n \in \mathbf{Z}} \xi_n z^{-n} = \frac{1}{\varepsilon} \exp \left(- \sum_{n \neq 0} \alpha_n q^{-|n|/2} z^{-n} \right) \\ &= \frac{1}{\varepsilon} \frac{\tau_-(zq^{1/2}) \tau_+(zq^{-1/2})}{\tau_-(zq^{-1/2}) \tau_+(zq^{1/2})}.\end{aligned}\quad (6)$$

The Poisson brackets among our dependent variables can be calculated as follows.

Lemma 1.1. *We have*

$$\{\eta(z), \eta(w)\} = \eta(z)\eta(w) \sum_{l \neq 0} \operatorname{sgn}(l)(1 - q^{|l|}) \left(\frac{w}{z}\right)^l, \quad (7)$$

$$\{\xi(z), \xi(w)\} = \xi(z)\xi(w) \sum_{l \neq 0} \operatorname{sgn}(l)(q^{-|l|} - 1) \left(\frac{w}{z}\right)^l, \quad (8)$$

$$\begin{aligned}\{\eta(z), \xi(w)\} &= \delta(q^{1/2}w/z) \frac{\tau_+(zq)\tau_+(z/q)}{\tau_+(z)\tau_+(z)} \\ &\quad - \delta(q^{-1/2}w/z) \frac{\tau_-(zq)\tau_-(z/q)}{\tau_-(z)\tau_-(z)},\end{aligned}\quad (9)$$

$$\{\eta(w), \tau_-(z)\} = \eta(w)\tau_-(z) \sum_{n>0} \left(\frac{w}{z}\right)^n, \quad (10)$$

$$\{\eta(w), \tau_+(z)\} = -\eta(w)\tau_+(z) \sum_{n>0} \left(\frac{z}{w}\right)^n, \quad (11)$$

$$\{\xi(w), \tau_-(z)\} = -\xi(w)\tau_-(z) \sum_{n>0} q^{-n/2} \left(\frac{w}{z}\right)^n, \quad (12)$$

$$\{\xi(w), \tau_+(z)\} = \xi(w)\tau_+(z) \sum_{n>0} q^{-n/2} \left(\frac{z}{w}\right)^n. \quad (13)$$

where $\delta(z) = \sum_{n \in \mathbf{Z}} z^n$.

Remark 1.1. In Ref. 9, a deep connection was found between the Macdonald Polynomials¹⁰ $P_\lambda(x; q, t)$ and the level one representation of the quantum algebra of Ding-Iohara $\mathcal{U}(q, t)$. We note that the α_n , $\eta(z)$ and $\xi(z)$ in the present paper are the level one generators of the Ding-Iohara algebra in the classical (namely commutative) limit given by letting the parameter as $t \rightarrow 1$. (Note that t here is one of the two parameters q, t for the Macdonald polynomials and shold not be confused with the time t .)

Proposition 1.1. *We have $\partial_t \eta(z) = \{\eta_0, \eta(z)\}$. Hence we identify η_0 with our Hamiltonian corresponding to the time t .*

Proof. This follows from (2) and (7) in Lamma 1.1. \square

Proposition 1.2. *We have $\{\eta_0, \xi_0\} = 0$.*

Proof. This follows from (9) in Lamma 1.1. \square

Remark 1.2. Since our Poisson algebra is the classical limit of the Ding-Iohara algebra, we have two sets of mutually Poisson commutative families, having η_0 and ξ_0 respectively, in the same way as was discussed in Ref. 9. As for the explicit form of them, see (40) and (39) below.

Because of the commutativity $\{\eta_0, \xi_0\} = 0$, we may interpret ξ_0 as another Hamiltonian corresponding to time \bar{t} .

Definition 1.2. Define $\partial_{\bar{t}}* = \{*, \xi_0\}$.

Proposition 1.3. *From (9) we have*

$$\partial_{\bar{t}}\eta(z) = \frac{\tau_+(zq)\tau_+(z/q)}{\tau_+(z)\tau_+(z)} - \frac{\tau_-(zq)\tau_-(z/q)}{\tau_-(z)\tau_-(z)}, \quad (14)$$

Now we turn to the equations for the tau-functions $\tau_{\pm}(z)$.

Proposition 1.4. *From (10), (11), (12) and (13), we have*

$$\begin{aligned} \partial_t\tau_-(z) &= \eta_-(z)\tau_-(z), & \partial_t\tau_+(z) &= -\eta_+(z)\tau_-(z), \\ \partial_{\bar{t}}\tau_-(zq^{-1/2}) &= \xi_-(z)\tau_-(zq^{-1/2}), & \partial_{\bar{t}}\tau_+(zq^{1/2}) &= -\xi_+(z)\tau_-(zq^{1/2}), \end{aligned} \quad (15)$$

where $\xi_+(z) = \sum_{n>0} \xi_{-n} z^n$ and $\xi_-(z) = \sum_{n>0} \xi_n z^{-n}$.

Suitable combinations of these may give us equations written in terms of the Hirota derivatives.

Definition 1.3. Define the Hirota derivative D_{t_1}, D_{t_2}, \dots by

$$\begin{aligned} &\left(D_{t_1}^{k_1} D_{t_2}^{k_2} \cdots \right) f \cdot g \\ &= \partial_{a_1}^{k_1} \partial_{a_2}^{k_2} \cdots f(t_1 + a_1, t_2 + a_2, \dots) g(t_1 - a_1, t_2 - a_2, \dots) \Big|_{a_1=a_2=\dots=0}. \end{aligned} \quad (16)$$

Proposition 1.5. *We have the Hirota equations*

$$D_t \tau_-(z) \cdot \tau_+(z) = \varepsilon \tau_-(z/q) \tau_+(zq) - \eta_0 \tau_-(z) \tau_+(z), \quad (17)$$

$$D_{\bar{t}} \tau_-(zq^{-1/2}) \cdot \tau_+(zq^{1/2}) \quad (18)$$

$$= \varepsilon^{-1} \tau_-(zq^{1/2}) \tau_+(zq^{-1/2}) - \xi_0 \tau_-(zq^{-1/2}) \tau_+(zq^{1/2}),$$

$$\begin{aligned} & \frac{1}{2} D_t D_{\bar{t}} \tau_{\pm}(z) \cdot \tau_{\pm}(z) \\ & + \tau_{\pm}(zq) \cdot \tau_{\pm}(z/q) - \tau_{\pm}(z) \cdot \tau_{\pm}(z) = 0. \end{aligned} \quad (19)$$

Proof. From (15), we have $\partial_t \tau_-(z) \cdot \tau_+(z) - \tau_-(z) \cdot \partial_t \tau_+(z) = (\eta(z) - \eta_0) \tau_-(z) \tau_+(z)$. Using (5) we have (17). The equation (18) can be derived in the same way. Eq. (19) is obtained from (14) and (15). \square

Remark 1.3. Note that Eq. (19) is nothing but the Toda field equation written in terms of the tau-function.⁵

One finds that the Heisenberg generators correspond to the standard dependent variables of the Toda field theory .

Definition 1.4. Set

$$\phi_+(z) = \sum_{n>0} \alpha_{-n} z^n, \quad \phi_-(z) = - \sum_{n>0} \alpha_n z^{-n}. \quad (20)$$

Proposition 1.6. *The $\phi_{\pm}(z)$ satisfy the Toda field equation*

$$\partial_t \partial_{\bar{t}} \phi_{\pm}(z) = e^{\phi_{\pm}(z) - \phi_{\pm}(z/q)} - e^{\phi_{\pm}(zq) - \phi_{\pm}(z)}. \quad (21)$$

Proof. This follows from Eqs. (3) and (9). \square

Motivated by the appearance of the Toda field equation (21), in this article we will try to understand how the 2D Toda hierarchy appears from the point of view of the Hamiltonian structure.

The present paper is organized as follows. In Section 2, we recall the Hirota-Miwa equation^{11,12} (22) for the 2D Toda hierarchy. Based on the n -soliton solutions, we derive some variants of the bilinear equations (Proposition 2.1). In Section 3, two sets of integrals of motion M_1, M_2, \dots and $\overline{M}_1, \overline{M}_2, \dots$ are introduced (Definition 3.2). Since at present we lack enough technologies to handle the evolution equations in general, we need to restrict ourself to some low lying cases. To show some evidences of the agreement, we check up to certain degree that exactly the same equations are obtained both from Proposition 2.1 and the Hamiltonian M_k 's. Our observation is summarized in Conjecture 3.2.

2. Hirota-Miwa equation for 2D Toda hierarchy

2.1. Hirota-Miwa equation

We briefly recall the Hirota-Miwa equation^{11,12} for the 2D Toda hierarchy.⁵ Using the n -soliton solution, we derive several variants of bilinear equations which can be connected to Eqs. (17), (18) and (19) obtained in the previous section.

Let $s, t = (t_1, t_2, \dots)$ and $\bar{t} = (\bar{t}_1, \bar{t}_2, \dots)$ be independent variables, and $\tau(s, t, \bar{t})$ be the tau-function of the 2D Toda hierarchy. For a parameter λ , we use the standard notation for the infinite vector $[\lambda] = (\lambda, \frac{1}{2}\lambda^2, \frac{1}{3}\lambda^3, \dots)$.

The Hirota-Miwa equation for the 2D Toda hierarchy is written as follows.

$$(1 - \alpha\beta)\tau(s, t, \bar{t})\tau(s, t + [\alpha], \bar{t} + [\beta]) - \tau(s, t + [\alpha], \bar{t})\tau(s, t, \bar{t} + [\beta]) \\ + \alpha\beta\tau(s + 1, t + [\alpha], \bar{t})\tau(s - 1, t, \bar{t} + [\beta]) = 0. \quad (22)$$

It is well known that the n -soliton solution to the Hirota-Miwa equation is given by

$$\tau(s, t, \bar{t}) = \sum_{r=0}^n \sum_{\substack{I \subset \{1, 2, \dots, n\} \\ |I|=r}} \prod_{\substack{\{i, j\} \subset I \\ i < j}} \frac{(\lambda_i - \lambda_j)(\mu_i - \mu_j)}{(\lambda_i - \mu_j)(\mu_i - \lambda_j)} \\ \times \prod_{k \in I} (\lambda_k / \mu_k)^s e^{\sum_{i=1}^{\infty} (t_i \lambda_k^i + \bar{t}_i \lambda_k^{-i}) - \sum_{i=1}^{\infty} (t_i \mu_k^i + \bar{t}_i \mu_k^{-i})}. \quad (23)$$

Let a_1, \dots, a_n be parameters. Set $\lambda_k = a_k, \mu_k = qa_k$ for $k = 1, 2, \dots, n$. Write $z = q^{-s}$. Then we have $(\lambda_k / \mu_k)^s = q^{-s} = z$.

We define $\tau_+(z, t, \bar{t})$ by the n -soliton solution $\tau(s, t, \bar{t})$ of 2D Toda hierarchy under this specialization ($\lambda_k = a_k, \mu_k = qa_k$).

Definition 2.1. Set

$$\tau_+(z, t, \bar{t}) = \sum_{r=0}^n z^r \sum_{\substack{I \subset \{1, 2, \dots, n\} \\ |I|=r}} \prod_{\substack{\{i, j\} \subset I \\ i < j}} \frac{(a_i - a_j)^2}{(a_i - qa_j)(a_i - q^{-1}a_j)} \\ \times \prod_{k \in I} e^{\sum_{i=1}^{\infty} (1 - q^i)t_i a_k^i + \sum_{i=1}^{\infty} (1 - q^{-i})\bar{t}_i a_k^{-i}}. \quad (24)$$

Note that $\tau_+(z, t, \bar{t})$ is a polynomial in z whose degree is n .

To introduce $\tau_-(z, t, \bar{t})$, we need a Lemma.

Lemma 2.1. *We have*

$$\begin{aligned}
& \tau_+(z, t, \bar{t} - [\beta]) \\
&= \sum_{r=0}^n z^r \sum_{\substack{I \subset \{1, 2, \dots, n\} \\ |I|=r}} \prod_{\substack{\{i, j\} \subset I \\ i < j}} \frac{(a_i - a_j)^2}{(a_i - qa_j)(a_i - q^{-1}a_j)} \\
&\quad \times \prod_{k \in I} \frac{1 - \beta/a_k}{1 - \beta/qa_k} e^{\sum_{i=1}^{\infty} (1-q^i)t_i a_k^i + \sum_{i=1}^{\infty} (1-q^{-i})\bar{t}_i a_k^{-i}} \\
&= z^n \prod_{1 \leq i < j \leq n} \frac{(a_i - a_j)^2}{(a_i - qa_j)(a_i - q^{-1}a_j)} \\
&\quad \times \prod_{k=1}^n \frac{1 - \beta/a_k}{1 - \beta/qa_k} e^{\sum_{i=1}^{\infty} (1-q^i)t_i a_k^i + \sum_{i=1}^{\infty} (1-q^{-i})\bar{t}_i a_k^{-i}} \\
&\quad \times \sum_{r=0}^n z^{-r} \sum_{\substack{I \subset \{1, 2, \dots, n\} \\ |I|=r}} \prod_{\substack{\{i, j\} \subset I \\ i < j}} \frac{(a_i - a_j)^2}{(a_i - qa_j)(a_i - q^{-1}a_j)} \\
&\quad \times \prod_{k \in I} d_k(\beta) e^{-\sum_{i=1}^{\infty} (1-q^i)t_i a_k^i - \sum_{i=1}^{\infty} (1-q^{-i})\bar{t}_i a_k^{-i}},
\end{aligned}$$

where

$$d_k(\beta) = \frac{1 - \beta/qa_k \varepsilon}{1 - \beta/a_k} \prod_{\substack{j \neq k \\ 1 \leq j \leq n}} \frac{(a_k - qa_j)(a_k - q^{-1}a_j)}{(a_k - a_j)^2}.$$

Now we define $\tau_-(z, t, \bar{t})$ by the following Laurent polynomial.

Definition 2.2. Set

$$\begin{aligned}
& \tau_-(z, t, \bar{t}) \\
&= \tau_+(z, t, \bar{t} - [q^n \varepsilon]) \times z^{-n} \prod_{1 \leq i < j \leq n} \frac{(a_i - qa_j)(a_i - q^{-1}a_j)}{(a_i - a_j)^2} \quad (25)
\end{aligned}$$

$$\begin{aligned}
& \times \prod_{k=1}^n \frac{1 - q^{n-1}\varepsilon/a_k}{1 - q^n\varepsilon/qa_k} e^{-\sum_{i=1}^{\infty} (1-q^i)t_i a_k^i - \sum_{i=1}^{\infty} (1-q^{-i})\bar{t}_i a_k^{-i}} \\
&= \sum_{r=0}^n z^{-r} \sum_{\substack{I \subset \{1, 2, \dots, n\} \\ |I|=r}} \prod_{\substack{\{i, j\} \subset I \\ i < j}} \frac{(a_i - a_j)^2}{(a_i - qa_j)(a_i - q^{-1}a_j)} \quad (26) \\
&\quad \times \prod_{k \in I} d_k(q^n \varepsilon) e^{-\sum_{i=1}^{\infty} (1-q^i)t_i a_k^i - \sum_{i=1}^{\infty} (1-q^{-i})\bar{t}_i a_k^{-i}}.
\end{aligned}$$

Proposition 2.1. *We have*

$$\begin{aligned} & \tau_-(z, t + [\alpha], \bar{t}) \tau_+(z, t, \bar{t}) \\ &= (1 - \alpha q^n \varepsilon) \prod_{k=1}^n \frac{(1 - \alpha a_k)}{(1 - \alpha q a_k)} \tau_-(z, t, \bar{t}) \tau_+(z, t + [\alpha], \bar{t}) \end{aligned} \quad (27)$$

$$+ \alpha \varepsilon \tau_-(z/q, t + [\alpha], \bar{t}) \tau_+(zq, t, \bar{t}),$$

$$\begin{aligned} & \tau_-(z/q, t, \bar{t} + [\beta]) \tau_+(z, t, \bar{t}) \\ &= (1 - \beta/q^n \varepsilon) \prod_{k=1}^n \frac{(1 - \beta/a_k)}{(1 - \beta/q a_k)} \tau_-(z/q, t, \bar{t}) \tau_+(z, t, \bar{t} + [\beta]) \end{aligned} \quad (28)$$

$$+ (\beta/\varepsilon) \tau_-(z, t, \bar{t} + [\beta]) \tau_+(z/q, t, \bar{t}),$$

$$\begin{aligned} & \tau_{\pm}(z, t + [\alpha], \bar{t}) \tau_{\pm}(z, t, \bar{t} + [\beta]) \\ &= (1 - \alpha \beta) \tau_{\pm}(z, t, \bar{t}) \tau_{\pm}(z, t + [\alpha], \bar{t} + [\beta]) \\ &+ \alpha \beta \tau_{\pm}(z/q, t + [\alpha], \bar{t}) \tau_{\pm}(zq, t, \bar{t} + [\beta]) = 0. \end{aligned} \quad (29)$$

Proof. Eq. (27) follows from the Hirota-Miwa equation (22), (24), (26) and Lemma 2.1. Noting that we have $\tau(s, t - [\alpha], \bar{t}) = \tau(s + 1, t, \bar{t} - [\alpha^{-1}])$, we have (28) in the same way. Eq. (29) follows from Eq. (22). \square

Remark 2.1. Note taht we may write

$$(1 - \alpha q^n \varepsilon) \prod_{k=1}^n \frac{(1 - \alpha a_k)}{(1 - \alpha q a_k)} = \exp \left(- \sum_{i=1}^{\infty} M_i \alpha^i \right), \quad (30)$$

$$(1 - \beta/q^n \varepsilon) \prod_{k=1}^n \frac{(1 - \beta/a_k)}{(1 - \beta/q a_k)} = \exp \left(- \sum_{i=1}^{\infty} \overline{M}_i \beta^i \right), \quad (31)$$

$$M_i = \frac{1 - q^i}{i} \left(a_1^i + \cdots + a_n^i + q^{ni} \varepsilon^i + q^{(n+1)i} \varepsilon^i + q^{(n+2)i} \varepsilon^i + \cdots \right), \quad (32)$$

$$\overline{M}_i = \frac{1 - q^{-i}}{i} \left(a_1^{-i} + \cdots + a_n^{-i} + q^{-ni} \varepsilon^{-i} + q^{-(n+1)i} \varepsilon^{-i} + \cdots \right). \quad (33)$$

Proposition 2.2. *By expanding (27) in α , we have the Hirota equations*

$$(D_{t_1} + M_1) \tau_-(z) \cdot \tau_+(z) = \varepsilon \tau_-(z/q) \cdot \tau_+(zq), \quad (34)$$

$$(D_{t_2} + 2M_2) \tau_-(z) \cdot \tau_+(z) = \varepsilon (D_{t_1} + M_1) \tau_-(z/q) \cdot \tau_+(zq), \quad (35)$$

$$\begin{aligned} & (D_{t_3} + 3M_3) \tau_-(z) \cdot \tau_+(z) + \frac{1}{8} (D_{t_1} + M_1)^3 \tau_-(z) \cdot \tau_+(z) \\ &= \frac{3}{4} \varepsilon (D_{t_2} + 2M_2) \tau_-(z/q) \cdot \tau_+(zq) + \frac{3}{8} \varepsilon (D_{t_1} + M_1)^2 \tau_-(z/q) \cdot \tau_+(zq), \end{aligned} \quad (36)$$

and so on. Here M_i 's are defined in (32).

Thus we found that Eq. (17) coincide with Eq. (34) under the identification $t = t_1, \bar{t} = \bar{t}_1$. Eqs. (18) and (19) coincide with the first nontrivial equation from Eqs. (28) and (29) respectively.

In the next section, we will check that Eqs. (35) and Eq. (36) also agree with the equations derived from the Poisson structure.

3. Poisson algebra and 2D Toda hierarchy

3.1. elementary and power sum symmetric functions

We need some facts about the symmetric functions.¹⁰ Let $x = (x_1, x_2, \dots)$ be an infinite set of independent indeterminates. Let $e_n(x)$ be the n -th elementary symmetric function, and $p_n(x)$ be the n -th power sum function. The generating functions for them are given by $E(y) = \sum_{n=0}^{\infty} e_n(x)y^n = \prod_{i=1}^{\infty}(1 + x_iy)$, and $P(y) = \sum_{n=1}^{\infty} \frac{1}{n} p_n(x)y^n = -\log E(-y)$. Solving the equation $P'(y) = -E'(-y)/E(-y)$, we have

$$p_n = \begin{vmatrix} e_1 & 1 & 0 & \cdots & 0 \\ 2e_2 & e_1 & 1 & \cdots & \\ 3e_3 & e_2 & e_1 & \cdots & \\ \vdots & & & \ddots & \vdots \\ ne_n & e_{n-1} & e_{n-2} & \cdots & e_1 \end{vmatrix}. \quad (37)$$

3.2. Integrals of motion from Ding-Iohara algebra

First we introduce some notations. We denote the constant term f_0 of a series $f(z) = \sum_{n \in \mathbf{Z}} f_n z^n$ by $[f(z)]_1$. We also use the same symbol for the case of a series with several variables. For examples, by $[f(z_1, z_2)]_1$ we denote the constant term $f_{0,0}$ of the series $f(z_1, z_2) = \sum_{n_1, n_2 \in \mathbf{Z}} f_{n_1, n_2} z_1^{n_1} z_2^{n_2}$.

Definition 3.1. Define the integrals of motion by

$$I_k = \left[\prod_{1 \leq i < j \leq k} \frac{1 - w_j/w_i}{1 - qw_j/w_i} \eta(w_1)\eta(w_2) \cdots \eta(w_k) \right]_1, \quad (38)$$

$$\bar{I}_k = \left[\prod_{1 \leq i < j \leq k} \frac{1 - w_j/w_i}{1 - q^{-1}w_j/w_i} \xi(w_1)\xi(w_2) \cdots \xi(w_k) \right]_1, \quad (39)$$

where the rational factors in w_i 's should be understood in the sense of the series as $(1 - w_j/w_i)/(1 - q^{\pm 1}w_j/w_i) = 1 + (1 - q^{\mp 1}) \sum_{n>0} (q^{\pm 1}w_j/w_i)^n$.

For example, we have $I_1 = \eta_0$ and $I_2 = \eta_0^2 + (1 - q^{-1}) \sum_{n>0} q^n \eta_{-n} \eta_n$, and so on. Based on the argument given in Ref. 9 with considering the classical limit ($t \rightarrow 1$), one can prove the following.

Proposition 3.1. *We have the commutativity $\{I_k, I_l\} = 0$, $\{\bar{I}_k, \bar{I}_l\} = 0$ and $\{I_k, \bar{I}_l\} = 0$.*

3.3. Integrals of motion associated with t and \bar{t}

Some explicit calculations show us that the integrals I_k and \bar{I}_k does not correspond to the Toda times t_1, t_2, \dots and $\bar{t}_1, \bar{t}_2, \dots$ in general. Hence our task is to find a suitable set of integrals, which we call M_k and \bar{M}_k .

At present, unfortunately, it is not easy to do the task purely within the framework of Poisson algebra. However, with the knowledge of the values of the integrals I_k and \bar{I}_k on the n -soliton solution (24), (26), we can guess the correct formula.

Conjecture 3.1. *Let $\tau_{\pm}(z, t, \bar{t})$ be as in (24) and (26). The quantities I_k 's and \bar{I}_k 's are independent of t and \bar{t} . The values are given by the following specialization of the elementary symmetric functions as*

$$\begin{aligned} I_k &= q^{-k(k-1)/2} (1-q)(1-q^2)\cdots(1-q^k) \\ &\quad \times e_k(a_1, \dots, a_n, q^n \varepsilon, q^{n+1} \varepsilon, \dots), \end{aligned} \tag{40}$$

$$\begin{aligned} \bar{I}_k &= q^{k(k-1)/2} (1-q^{-1})(1-q^{-2})\cdots(1-q^{-k}) \\ &\quad \times e_k(a_1^{-1}, \dots, a_n^{-1}, q^{-n} \varepsilon^{-1}, q^{-n-1} \varepsilon^{-1}, \dots). \end{aligned} \tag{41}$$

As for the statement about I_k in Eq. (40), see Ref. 2.

Remark 3.1. For small k , we have

$$\begin{aligned} I_1 &= (1-q)(a_1 + \cdots + a_n) + q^n \varepsilon, \\ I_2 &= q^{-1}(1-q)(1-q^2)(a_1 a_2 + a_1 a_3 + \cdots + a_{n-1} a_n) \\ &\quad + q^{n-1}(1-q^2)(a_1 + \cdots + a_n) \varepsilon + q^{2n} \varepsilon^2, \end{aligned}$$

and so on.

Definition 3.2. Set $I'_k = q^{k(k-1)/2} ((1-q)(1-q^2)\cdots(1-q^k))^{-1} I_k$ and $\bar{I}'_k = q^{-k(k-1)/2} ((1-q^{-1})(1-q^{-2})\cdots(1-q^{-k}))^{-1} \bar{I}_k$ for simplicity of display.

Define

$$M_k = \frac{1-q^k}{k} \begin{vmatrix} I'_1 & 1 & 0 & \cdots & 0 \\ 2I'_2 & I'_1 & 1 & \cdots & \\ 3I'_3 & I'_2 & I'_1 & \cdots & \\ \vdots & & & \ddots & \vdots \\ kI'_k & I'_{k-1} & I'_{k-2} & \cdots & I'_1 \end{vmatrix}, \quad (42)$$

$$\overline{M}_k = \frac{1-q^{-k}}{k} \begin{vmatrix} \overline{I}'_1 & 1 & 0 & \cdots & 0 \\ 2\overline{I}'_2 & \overline{I}'_1 & 1 & \cdots & \\ 3\overline{I}'_3 & \overline{I}'_2 & \overline{I}'_1 & \cdots & \\ \vdots & & & \ddots & \vdots \\ k\overline{I}'_k & \overline{I}'_{k-1} & \overline{I}'_{k-2} & \cdots & \overline{I}'_1 \end{vmatrix}. \quad (43)$$

Remark 3.2. Conjecture 3.1 implies that if $\tau_{\pm}(z, t, \bar{t})$ be as in (24) and (26), M_k 's and \overline{M}_k 's are given by

$$M_k = \frac{1-q^k}{k} p_k(a_1, \dots, a_n, q^n \varepsilon, q^{n+1} \varepsilon, \dots),$$

$$\overline{M}_k = \frac{1-q^{-k}}{k} p_k(a_1^{-1}, \dots, a_n^{-1}, q^{-n} \varepsilon^{-1}, q^{-n-1} \varepsilon^{-1}, \dots).$$

3.4. Formulas for M_2 and M_3

Now we come back to our study of the Poisson algebra to check the higher Hirota equations (35) and (36).

It is desirable to find some reasonably simple expressions for M_k 's. At present, however, we only have the following partial results.

Lemma 3.1. We have $M_1 = [\eta(w)]_1$ and

$$M_2 = \left[\left(\frac{1}{2} + \frac{qw_2/w_1}{1-qw_2/w_1} \right) \eta(w_1) \eta(w_2) \right]_1, \quad (44)$$

$$M_3 = \left[\left(\frac{1}{3} + \frac{qw_3/w_2}{(1-qw_2/w_1)(1-qw_3/w_2)} \right) \eta(w_1) \eta(w_2) \eta(w_3) \right]_1. \quad (45)$$

Remark 3.3. It is an open problem to find a simple expression as above for M_4, M_5, \dots .

3.5. Main conjecture and equations with respect to t_2, t_3

Definition 3.3. Set $\partial_{t_k} * = \{M_k, *\}$ and $\partial_{\bar{t}_k} * = \{*, \overline{M}_k\}$.

Now we are ready to state our conjecture.

Conjecture 3.2. *Calculating ∂_{t_k} and $\partial_{\bar{t}_k}$ by using the Poisson brackets given in Definition 3.3, we recover the same equation derived from the Hirota-Miwa equations (27), (28) and (29).*

The rest of the paper is devoted to give some evidence of our conjecture.

Proposition 3.2. *Calculating ∂_{t_2} and ∂_{t_1} by using the Poisson brackets given in Definition 3.3, we recover the same equation as in Eq. (35).*

Proof. From (10), (11) and (44), we have

$$\begin{aligned} & \frac{\partial_{t_2}\tau_-(z)}{\tau_-(z)} - \frac{\partial_{t_2}\tau_+(z)}{\tau_+(z)} + 2M_2 \\ &= \left[(\delta(w_1/z) + \delta(w_2/z)) \left(\frac{1}{2} + \frac{qw_2/w_1}{1-qw_2/w_1} \right) \eta(w_1)\eta(w_2) \right]_{1,w_1,w_2} \\ &= M_1\eta(z) + \eta(z) \left[\frac{qw_2/z}{1-qw_2/z} \eta(w_2) \right]_{1,w_2} + \eta(z) \left[\frac{qz/w_1}{1-qz/w_1} \eta(w_1) \right]_{1,w_1} \\ &= M_1\eta(z) + \eta(z)\eta_-(z/q) + \eta(z)\eta_+(zq). \end{aligned}$$

Using (5) and (15) (with $t = t_1$), we have the result. \square

Finally, we study the Hirota equation involving the third time t_3 .

Proposition 3.3. *Calculating ∂_{t_3} , ∂_{t_2} and ∂_{t_1} by using the Poisson brackets given in Definition 3.3, we have*

$$\begin{aligned} & (D_{t_3} + 3M_3) \tau_-(z) \cdot \tau_+(z) \tag{46} \\ &= \frac{1}{2}\varepsilon (D_{t_2} + 2M_2) \tau_-(z/q) \cdot \tau_+(zq) + \frac{1}{2}\varepsilon (D_{t_1} + M_1)^2 \tau_-(z/q) \cdot \tau_+(zq), \end{aligned}$$

$$\begin{aligned} & (D_{t_1} + M_1)^3 \tau_-(z) \cdot \tau_+(z) \tag{47} \\ &= 2\varepsilon (D_{t_2} + 2M_2) \tau_-(z/q) \cdot \tau_+(zq) - \varepsilon (D_{t_1} + M_1)^2 \tau_-(z/q) \cdot \tau_+(zq). \end{aligned}$$

Corollary 3.1. *We recover (36) from the the Hamiltonian structure.*

Proof of Proposition 3.3. They follow from Lemmas 3.2, 3.3, 3.4 and 3.5 below. \square

Lemma 3.2. *From (10), (11) and (45), we have*

$$\begin{aligned} & \frac{(D_{t_3} + 3M_3)\tau_-(z) \cdot \tau_+(z)}{\tau_-(z) \cdot \tau_+(z)} \\ &= \eta(z) \left(M_2 + \frac{1}{2}M_1^2 + M_1(\eta_+(zq) + \eta_-(z/q)) + \eta_+(zq)\eta_-(z/q) \right) \\ &+ \eta(z) \left[\left(\frac{qw_1/w_2}{1-qw_1/w_2} \frac{qw_2/z}{1-qw_2/z} \right. \right. \\ &\quad \left. \left. + \frac{qw_2/w_1}{1-qw_2/w_1} \frac{qz/w_2}{1-qz/w_2} \right) \eta(w_1)\eta(w_2) \right]_{1,w_1,w_2}. \end{aligned}$$

Lemma 3.3. *From (7), (10) and (11), we have*

$$\begin{aligned} & \frac{(D_{t_1} + M_1)^3\tau_-(z) \cdot \tau_+(z)}{\tau_-(z) \cdot \tau_+(z)} \\ &= \eta(z) \left(4M_2 - M_1^2 + M_1(\eta_+(zq) + \eta_-(z/q)) - 2\eta_+(zq)\eta_-(z/q) \right) \\ &+ 2\eta(z) \left[\left(\frac{qw_1/w_2}{1-qw_1/w_2} + \frac{qw_2/w_1}{1-qw_2/w_1} \right) \right. \\ &\quad \times \left. \left(\frac{qw_2/z}{1-qw_2/z} + \frac{qz/w_2}{1-qz/w_2} \right) \eta(w_1)\eta(w_2) \right]_{1,w_1,w_2} \\ &- \eta(z) \left[\left(\frac{qw_1/w_2}{1-qw_1/w_2} - \frac{qw_2/w_1}{1-qw_2/w_1} \right) \right. \\ &\quad \times \left. \left(\frac{qw_2/z}{1-qw_2/z} - \frac{qz/w_2}{1-qz/w_2} \right) \eta(w_1)\eta(w_2) \right]_{1,w_1,w_2}. \end{aligned}$$

Lemma 3.4. *From (10), (11) and (44), we have*

$$\begin{aligned} & \frac{(D_{t_2} + 2M_2)\tau_-(z/q) \cdot \tau_+(zq)}{\tau_-(z/q) \cdot \tau_+(zq)} \\ &= 2M_2 + M_1(\eta_+(zq) + \eta_-(z/q)) \\ &+ \left[\left(\frac{qw_1/w_2}{1-qw_1/w_2} + \frac{qw_2/w_1}{1-qw_2/w_1} \right) \right. \\ &\quad \times \left. \left(\frac{qw_2/z}{1-qw_2/z} + \frac{qz/w_2}{1-qz/w_2} \right) \eta(w_1)\eta(w_2) \right]_{1,w_1,w_2}. \end{aligned}$$

Lemma 3.5. *From (7), (10) and (11), we have*

$$\begin{aligned} & \frac{(D_{t_1} + M_1)^2 \tau_-(z/q) \cdot \tau_+(zq)}{\tau_-(z/q) \cdot \tau_+(zq)} \\ &= M_1^2 + M_1(\eta_+(zq) + \eta_-(z/q)) + 2\eta_+(zq)\eta_-(z/q) \\ &+ \left[\left(\frac{qw_1/w_2}{1-qw_1/w_2} - \frac{qw_2/w_1}{1-qw_2/w_1} \right) \right. \\ &\quad \left. \times \left(\frac{qw_2/z}{1-qw_2/z} - \frac{qz/w_2}{1-qz/w_2} \right) \eta(w_1)\eta(w_2) \right]_{1,w_1,w_2}. \end{aligned}$$

Acknowledgments

The authors thank Kenji Kajiwara, Paul Wiegmann and Masatoshi Noumi for stimulating discussions. YT is supported by The Promotion and Mutual Aid Corporation for Private Schools of Japan.

References

1. J. Shiraishi and Y. Tutiya, Periodic ILW equation with discrete Laplacian, *J. Phys. A* **42** (2009), no. 40, 404018, 15 pp.
2. Y. Tutiya and J. Shiraishi, On some special solutions to periodic Benjamin-Ono equation with discrete Laplacian, arXiv:0911.5005.
3. T.B. Benjamin, Internal waves of permanent form in fluids of great depth, *J. Fluid. Mech.*, **29** (1967), 559.
4. H. Ono, Algebraic solitary waves in stratified fluids, *J. Phys. Soc. Japan*, **30** (1975), 1082.
5. K. Ueno and K. Takasaki, Toda lattice hierarchy, Group Representations and Systems of Differential Equations, K. Okamoto (ed.), *Adv. Stud. Pure Math.* vol. **4**, pp.1-95 (North-Holland, Amsterdam, Kinokuniya, Tokyo, 1884).
6. P.M. Santini, Inverse Problems **5** (1989) 203.
7. D.R. Lebedev, A.O. Radul, *Commun. Math. Phys.* **91** (1983) 543.
8. A.G. Abanov, E. Bettelheim and P. Wiegmann, Integrable hydrodynamics of Calogero-Sutherland model: bidirectional Benjamin-Ono equation, *J. Phys. A: Math. Theor.* **42** (2009) 135201 (24pp).
9. B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi and S. Yanagida, A commutative algebra on degenerate \mathbb{CP}^1 and Macdonald polynomials, *J. Math. Phys.* **50** (2009), no. 9, 095215, 42 pp.
10. I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd edn. Oxford University Press, 1995.
11. R. Hirota, Discrete analogue of a generalized Toda equation, *J. Phys. Soc. Jpn.* **50** (1981) 3785-3791.
12. T. Miwa, On Hirota's difference equations, *Proc. Japan Acad.* **58** (1982) 9-12.